

5.6

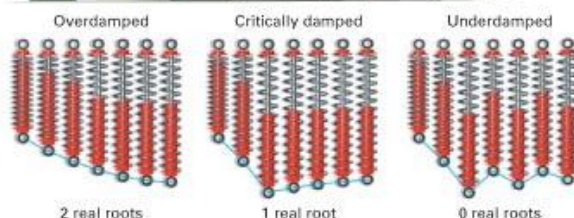
Quadratic Equations and Complex Numbers



Why Solutions to many real-world problems, such as classifying a shock absorber spring system, involve complex numbers.

Objectives

- Classify and find all roots of a quadratic equation.
- Graph and perform operations on complex numbers.



Each diagram represents the motion of a shock absorber spring over a period of 2 seconds from left to right.

APPLICATION ENGINEERING

Carlos and Keiko are mechanical engineers who are analyzing car shock absorbers. They know that the motion of the spring is affected by a damping force. Three situations are possible depending on how the mass of the car, the spring, and the damping force are related. The roots of $x^2 + mx + n = 0$, where m and n depend on the car's mass, the spring, and the damping force, help them to classify a spring system. *You will classify a spring system after Example 1.*

The Discriminant

When you apply the quadratic formula to any quadratic equation, you will find that the value of $b^2 - 4ac$ is either positive, negative, or 0. The expression $b^2 - 4ac$ is called the **discriminant** of a quadratic equation.

You can see from the quadratic formula that if $b^2 - 4ac > 0$, the formula will give two different answers. If $b^2 - 4ac = 0$, there will be one answer, called a **double root**. If $b^2 - 4ac < 0$, the radical will be undefined for real numbers, so the formula gives no real solutions.

Solutions of a Quadratic Equation

Let $ax^2 + bx + c = 0$, where $a \neq 0$.

- If $b^2 - 4ac > 0$, then the quadratic equation has 2 distinct real solutions.
- If $b^2 - 4ac = 0$, then the equation has 1 real solution, a double root.
- If $b^2 - 4ac < 0$, then the equation has 0 real solutions.

EXAMPLE 1 Find the discriminant for each equation. Then determine the number of real solutions for each equation by using the discriminant.

a. $2x^2 + 4x + 1 = 0$ b. $2x^2 + 4x + 2 = 0$ c. $2x^2 + 4x + 3 = 0$

SOLUTION

a. $2x^2 + 4x + 1 = 0$
 $b^2 - 4ac$
 $= 4^2 - 4(2)(1)$
 $= 8$

Because $b^2 - 4ac > 0$, the equation has 2 real solutions.

b. $2x^2 + 4x + 2 = 0$
 $b^2 - 4ac$
 $= 4^2 - 4(2)(2)$
 $= 0$

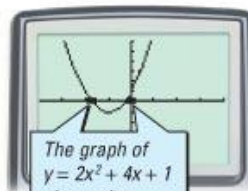
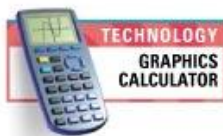
Because $b^2 - 4ac = 0$, the equation has 1 real solution.

c. $2x^2 + 4x + 3 = 0$
 $b^2 - 4ac$
 $= 4^2 - 4(2)(3)$
 $= -8$

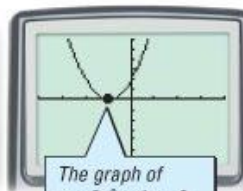
Because $b^2 - 4ac < 0$, the equation has no real solutions.

CHECK

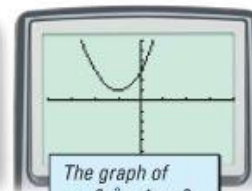
Graph the related functions to check.



The graph of $y = 2x^2 + 4x + 1$ shows that $2x^2 + 4x + 1 = 0$ for two different values of x .



The graph of $y = 2x^2 + 4x + 2$ shows that $2x^2 + 4x + 2 = 0$ for one value of x .



The graph of $y = 2x^2 + 4x + 3$ shows that $2x^2 + 4x + 3 = 0$ for no real x .

TRY THIS Identify the number of real solutions to $-3x^2 - 6x + 15 = 0$.

CHECKPOINT ✓ Refer to the shock-absorber situation described on page 314. Classify a spring system in which $m = 8$ and $n = 24$.

Complex Numbers

CULTURAL CONNECTION: EUROPE Whether it is possible to take the square root of a negative number is a question that puzzled mathematicians for a long time. In the sixteenth century, Italian mathematician Girolamo Cardano (1501–1576) was the first to use complex numbers to solve quadratic equations. Leonhard Euler (1707–1783) defined the **imaginary unit**, i , such that $i = \sqrt{-1}$ and $i^2 = -1$. In the nineteenth century, the theoretical basis of complex numbers was rigorously developed.

Imaginary Numbers

If $r > 0$, then the **imaginary number** $\sqrt{-r}$ is defined as follows:

$$\sqrt{-r} = \sqrt{-1} \cdot \sqrt{r} = i\sqrt{r}$$

For example, $\sqrt{-4} = \sqrt{-1} \cdot \sqrt{4} = 2i$ and $\sqrt{-6} = \sqrt{-1} \cdot \sqrt{6} = i\sqrt{6}$.

With the quadratic formula and the imaginary unit, i , you can find solutions to any quadratic equation. This is shown in Example 2.

EXAMPLE 2 Use the quadratic formula to solve $3x^2 - 7x + 5 = 0$.

SOLUTION

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(5)}}{2(3)} \quad \text{Substitute } a = 3, b = -7, \text{ and } c = 5.$$

$$x = \frac{7 \pm \sqrt{-11}}{6}$$

$$x = \frac{7}{6} + \frac{\sqrt{-11}}{6} \quad \text{or} \quad x = \frac{7}{6} - \frac{\sqrt{-11}}{6}$$

$$x = \frac{7}{6} + \frac{i\sqrt{11}}{6} \quad x = \frac{7}{6} - \frac{i\sqrt{11}}{6} \quad \text{Replace } \sqrt{-11} \text{ with } i\sqrt{11}.$$

The numbers $\frac{7}{6} + \frac{i\sqrt{11}}{6}$ and $\frac{7}{6} - \frac{i\sqrt{11}}{6}$ are complex numbers.

TRY THIS Use the quadratic formula to solve $-4x^2 + 5x - 3 = 0$.

Complex Numbers

A **complex number** is any number that can be written as $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$; a is called the **real part** and b is called the **imaginary part**.

The form $a + bi$ is called the *standard form* of a complex number. Real numbers are complex numbers for which $b = 0$. A complex number is called a *pure imaginary number* if its real part, a , is 0.

EXAMPLE 3 Find x and y such that $7x - 2iy = 14 + 6i$.

SOLUTION

Two complex numbers are equal if their real parts are equal and their imaginary parts are equal.

Real parts

$$7x = 14$$

$$x = 2$$

Imaginary parts

$$-2y = 6$$

$$y = -3$$

Thus, $x = 2$ and $y = -3$.

TRY THIS Find x and y such that $2x + 3iy = -8 + 10i$.

Operations With Complex Numbers

EXAMPLE 4 Find each sum or difference.

a. $(-3 + 5i) + (7 - 6i)$

b. $(-3 - 8i) - (-2 - 9i)$

SOLUTION

Add or subtract the corresponding real parts and imaginary parts.

a. $(-3 + 5i) + (7 - 6i)$

$$= (-3 + 7) + (5i - 6i)$$

$$= 4 - 1i$$

$$= 4 - i$$

b. $(-3 - 8i) - (-2 - 9i)$

$$= (-3 + 2) - (8i - 9i)$$

$$= -1 + 1i$$

$$= -1 + i$$

Two complex numbers whose real parts are opposites and whose imaginary parts are opposites are called *additive inverses*.

$$(4 + 3i) + (-4 - 3i) = 0 + 0i = 0$$

CHECKPOINT ✓ What is the additive inverse of $2i - 12$?

EXAMPLE 5 Multiply $(2 + i)(-5 - 3i)$.

SOLUTION

$$(2 + i)(-5 - 3i) = 2(-5 - 3i) + i(-5 - 3i)$$

$$= -10 - 6i - 5i - 3i^2$$

$$= -10 - 11i - 3(-1)$$

$$= -7 - 11i$$

Apply the Distributive Property.

Replace i^2 with -1 .

TRY THIS Multiply $(6 - 4i)(5 - 4i)$.

Activity

Exploring Powers of i

You will need: no special materials

1. Copy and complete the table below. (Hint: Recall that $i^2 = -1$.)

i	$i^2 =$	$i^3 =$	$i^4 =$
$i^5 =$	$i^6 =$	$i^7 =$	$i^8 =$

2. Observe the patterns in the table above. Use your observations to complete the table below.

$i^9 =$	$i^{10} =$	$i^{11} =$	$i^{12} =$
$i^{13} =$	$i^{14} =$	$i^{15} =$	$i^{16} =$

CHECKPOINT ✓ 3. Describe the pattern that occurs in the powers of i . Explain how to use the pattern to evaluate i^{41} , i^{66} , i^{75} , and i^{100} .

In order to simplify a fraction containing complex numbers, you often need to use the *conjugate of a complex number*. For example, the conjugate of $2 + 5i$ is $2 - 5i$ and the conjugate of $1 - 3i$ is $1 + 3i$.

Conjugate of a Complex Number

The **conjugate** of a complex number $a + bi$ is $a - bi$. The conjugate of $a + bi$ is denoted $\overline{a + bi}$.

To simplify a quotient with an imaginary number in the denominator, multiply by a fraction equal to 1, using the conjugate of the denominator, as shown in Example 6. This process is called **rationalizing the denominator**.

EXAMPLE 6 Simplify $\frac{2+5i}{2-3i}$. Write your answer in standard form.

SOLUTION

$$\frac{2+5i}{2-3i} = \frac{2+5i}{2-3i} \cdot \frac{2+3i}{2+3i} \quad \text{Multiply by 1, using the conjugate of the denominator.}$$

$$= \frac{(2+5i)(2+3i)}{(2-3i)(2+3i)}$$

$$= \frac{4+10i+6i+15i^2}{4-6i+6i-9i^2}$$

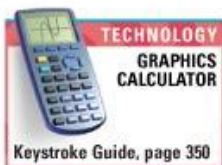
$$= \frac{-11+16i}{13}$$

$$= -\frac{11}{13} + \frac{16}{13}i$$

CHECK

In complex mode, enter the expression. Then express the answer with fractions.

Note that the last term is $\frac{16}{13}i$, not $\frac{16}{13}i$.



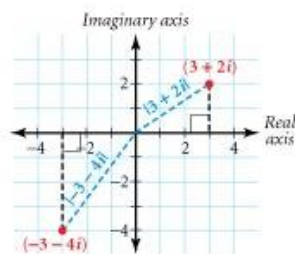
TRY THIS Simplify $\frac{3-4i}{2+i}$. Write your answer in standard form.

CRITICAL THINKING

Show that if a complex number is equal to its conjugate, the number is real.



Complex numbers are graphed in the *complex plane*. In the **complex plane**, the horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis**. To graph the complex number $a + bi$, plot the point (a, b) . For example, the point $(3, 2)$ represents the complex number $3 + 2i$ and the point $(-3, -4)$ represents the complex number $-3 - 4i$.

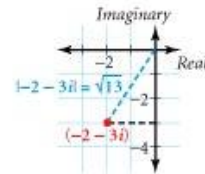


The absolute value of a real number is its distance from zero on the number line. Likewise, the **absolute value of a complex number** $a + bi$, denoted by $|a + bi|$, is its distance from the origin in the complex plane. By the Pythagorean Theorem, $|a + bi| = \sqrt{a^2 + b^2}$. For example, $|3 + 2i| = \sqrt{3^2 + 2^2} = \sqrt{13}$ and $|-3 - 4i| = \sqrt{(-3)^2 + (-4)^2} = 5$.

EXAMPLE 7 Evaluate $|-2 - 3i|$. Sketch a diagram that shows $-2 - 3i$ and $|-2 - 3i|$.

SOLUTION

$$\begin{aligned} |-2 - 3i| &= \sqrt{(-2)^2 + (-3)^2} \\ &= \sqrt{4 + 9} \\ &= \sqrt{13} \end{aligned}$$



TRY THIS Evaluate $|-3 + 5i|$. Sketch a diagram that shows $-3 + 5i$ and $|-3 + 5i|$.

CRITICAL THINKING The real numbers are said to be “well-ordered,” meaning that any real number is either larger or smaller than any other given real number. Are the complex numbers well-ordered? Explain.

Exercises

Communicate

1. What can the discriminant tell you about the solutions of a quadratic equation?
2. How do you simplify a rational expression that contains a complex number in the denominator?
3. How do you graph a real number in the complex plane? a pure imaginary number?

Guided Skills Practice

Determine the number of real solutions for each equation. (EXAMPLE 1)

4. $x^2 + 2x + 1 = 0$ 5. $2x^2 + 4x + 5 = 0$ 6. $x^2 + 3x + 1 = 0$

7. Solve the equation $2x^2 + 5x + 4 = 0$. (EXAMPLE 2)

8. Find x and y such that $-2x + 3yi = 2 + 6i$. (EXAMPLE 3)

Simplify each expression.

9. $(2 + 3i) + (4 + 7i)$ (EXAMPLE 4)
10. $(8 + 4i) - (3 + 2i)$ (EXAMPLE 4)
11. $(-1 + 2i)(3 + 4i)$ (EXAMPLE 5)
12. $\frac{-1 + 4i}{2 + 3i}$ (EXAMPLE 6)
13. Evaluate $|3 + 4i|$. Sketch a diagram that shows $3 + 4i$ and $|3 + 4i|$. (EXAMPLE 7)



Practice and Apply

Identify the real and imaginary parts of each complex number.

14. $-5 + 6i$ 15. $2 + i$ 16. 6 17. $4i$

Simplify.

18. $\sqrt{-36}$ 19. $\sqrt{-100}$ 20. $\sqrt{-13}$
 21. $\sqrt{-17}$ 22. $(-3i)^2$ 23. $(-7i)^2$

Find the discriminant, and determine the number of real solutions. Then solve.

24. $x^2 + 5x + 8 = 0$ 25. $3x^2 - 5x + 4 = 0$ 26. $x^2 - 7x = -10$
 27. $5x^2 - 5x + 2 = 0$ 28. $-x^2 + 8x - 19 = 0$ 29. $x^2 - 3x = 7$
 30. $-2x^2 + 10x = 15$ 31. $2x^2 - 6x = -5$ 32. $4x^2 + x - 2 = 0$
 33. $2x^2 + 3x = 0$ 34. $5x^2 + 4x = -5$ 35. $2x^2 + 2x + 2 = 0$
 36. $16 - 8x = -x^2$ 37. $x^2 + 49 = 14x$ 38. $-x^2 + 4x - 5 = 0$
 39. $8x^2 + 5x + 2 = 0$ 40. $4x^2 + 9 = 12x$ 41. $1 + 9x^2 = 6x$

Find x and y .

42. $6x + 7iy = 18 - 21i$ 43. $3x - 4iy = 4 + 4i$ 44. $2x + 5i = 8 + 20yi$

Perform the indicated addition or subtraction.

45. $(-2 + 3i) + (-1 - 4i)$ 46. $(1 + 2i) - (1 + 5i)$ 47. $\left(\frac{1}{2} + \frac{2}{5}i\right) + \left(\frac{1}{2} - \frac{1}{5}i\right)$
 48. $\left(\frac{3}{8} + \frac{2}{3}i\right) - \left(\frac{1}{4} - \frac{1}{3}i\right)$ 49. $(3 + i) + (6 - 2i)$ 50. $(8 - 6i) - (4 - 3i)$

Multiply.

51. $i(2 + i)$ 52. $(1 + i)(1 - i)$ 53. $(-5 - i)(-2 + 2i)$
 54. $(-5 + 3i)(2 - 3i)$ 55. $(6 - 7i)^2$ 56. $(2 - 4i)^2$
 57. $(-1 + i\sqrt{5})^2$ 58. $(2 + i\sqrt{3})^2$ 59. $(2 - 3i\sqrt{2})^2$

Write the conjugate of each complex number.

60. $2 + 3i$ 61. 8 62. $-4 - i$ 63. $8 - 3i$

Simplify.

64. $\frac{4 + 2i}{2 + i}$ 65. $\frac{3 + 2i}{5 + i}$ 66. $\left(\frac{1}{2} + i\right)\left(\frac{2}{3} + \frac{1}{4}i\right)$
 67. $\frac{4 + 2i}{\frac{2}{3} + \frac{1}{2}i}$ 68. $2(3 - 2i) + 5(1 + i)$ 69. $\frac{1}{4}(1 + i\sqrt{3})(1 - i\sqrt{3})$
 70. $(1 + i)^2 + \frac{2 + 2i}{2 + i}$ 71. $\frac{3 + i}{4 + i} + 1 + i$ 72. $(-3i)(3i) - (2 + 2i)$
 73. $i(1 + i) - (3 + i)$ 74. $i^3(5 + i) + 7i$ 75. $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2$

Graph each number and its conjugate in the complex plane.

76. $5 + 3i$ 77. $-2 + 4i$ 78. $-3 - 5i$ 79. $2 - 3i$
 80. $-2i$ 81. $6i$ 82. $-2 + 7i$ 83. $-7 - 2i$
 84. -4 85. $5 + i$ 86. $3 - 5i$ 87. 3



Evaluate. Then sketch a diagram that shows the absolute value.

88. $|1 + i|$

89. $|i|$

90. $\left|\frac{3}{5} + \frac{4}{5}i\right|$

91. $|1 + 0.01i|$

92. $|2i|$

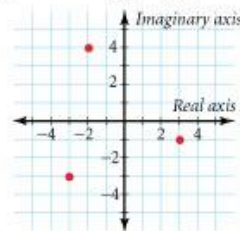
93. $|2 + 3i|$

94. $\left|\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right|$

95. $\left|\frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}i\right|$

96. Identify the complex numbers graphed on the complex plane at right.

97. Identify the coordinates of the conjugate of each complex number graphed at right.



CHALLENGES

98. If $\overline{c + di} = -(c + di)$, what can you say about the complex number $c + di$? (Hint: Solve for c .)

99. Let $a > 0$ and $b > 0$. Plot $a + bi$, $a - bi$, $-(a + bi)$, and $-a + bi$ in the complex plane. Connect the points with vertical and horizontal lines. What is the result?

CONNECTION

100. **TRANSFORMATIONS** Describe the relationship between the graph of a complex number and the graph of its conjugate as a translation, rotation, or reflection.

Look Back

Graph each function and its inverse on the same coordinate plane.

(LESSON 2.5)

101. $f(x) = 2x - 3$

102. $f(x) = -3x + 5$

103. $f(x) = 2x$

Let $f(x) = 3x - 5$. Find the indicated function. (LESSONS 2.4 AND 2.5)

104. $f^{-1}(x)$

105. $(f \circ f^{-1})(x)$

106. $(f^{-1} \circ f)(x)$

APPLICATIONS

107. **BUSINESS** Gene ordered 8 prints for his new restaurant. Each unframed print cost \$50, and each framed print cost \$98. The total cost of the prints was \$640. How many framed prints did Gene buy?

(LESSONS 3.1 AND 3.2)



108. **RECYCLING** A recycling center pays \$1.25 for 100 pounds of newspaper and \$0.40 for 1 pound of aluminum. Linda took a 460-pound load of aluminum and newspaper to the recycling center and was paid \$21.82. How many pounds of aluminum and how many pounds of newspaper did Linda have?

(LESSONS 3.1 AND 3.2)

Look Beyond

Predict the zeros and verify by graphing.

109. $f(x) = (x - 2)(x + 3)(x + 5)$

110. $f(x) = x(x - 6)(x + 1)$

111. $f(x) = x^3 + x^2 - 2x$

112. $f(x) = x^3 + 2x^2 + x$